

EQUIVARIANT BORDISM OF MAPS

BY

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ABSTRACT. This note computes the bordism classification of equivariant maps between closed manifolds with action of a cyclic group of prime order.

1. Introduction. Being given a finite group G , one would like to classify equivariant maps $f: (N^n, \phi) \rightarrow (M^m, \psi)$ between closed manifolds with G action. Roughly, this would combine the analysis of G actions introduced by Conner and Floyd [1] and the analysis of maps in [2].

This paper is intended to illustrate that the Conner and Floyd fixed point methods may be used to classify equivariant maps up to bordism. Attention will be restricted to the case of G cyclic of prime order, leaving open the generalization to bordism groups of equivariant maps needed for a full scale study. Notation will follow [3].

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2. Definition of the G bordism groups. Let G be a finite group. A family \mathcal{F} in G is a collection of subgroups of G so that $H \in \mathcal{F}$, $g \in G$ and $K \subset H$ imply $gHg^{-1} \in \mathcal{F}$ and $K \in \mathcal{F}$. Let $\mathcal{F}' \subset \mathcal{F}$ be a pair of families. An $(\mathcal{F}, \mathcal{F}')$ -free map of dimension (m, n) is a triple $((M^m, \psi), (N^n, \phi), f)$ where M^m and N^n are compact manifolds with boundary of dimensions m and n , $\psi: G \times M \rightarrow M$ and $\phi: G \times N \rightarrow N$ are differentiable G actions which are $(\mathcal{F}, \mathcal{F}')$ -free; i.e. each isotropy group G_x , $x \in M$ or N , belongs to \mathcal{F} , and each isotropy group G_x , $x \in \partial M$ or ∂N , belongs to \mathcal{F}' , and $f: (N, \partial N, \phi) \rightarrow (M, \partial M, \psi)$ is a differentiable map equivariant with respect to the given actions. Two $(\mathcal{F}, \mathcal{F}')$ -free maps of dimension (m, n) , $((M, \psi), (N, \phi), f)$ and $((M', \psi'), (N', \phi'), f')$ are equivalent if there is a triple $((V, V^+, \Psi), (W, W^+, \Phi), F)$ where V and W are compact manifolds with boundary; M, M', V^+ being regularly imbedded submanifolds of ∂V with $\partial V = M \cup M' \cup V^+$, $M \cap M' = \emptyset$, $V^+ \cap M = \partial M$, $V^+ \cap M' = \partial M'$, $V^+ \cap (M \cup M') = \partial V^+$; N, N', W^+ being similarly related in ∂W ; Ψ and Φ are differentiable G actions extending ψ, ψ', ϕ, ϕ' with V^+, W^+ being invariant, so that V and W are \mathcal{F} -free and V^+ and W^+ are \mathcal{F}' -free; and $F: (W, W^+, \Phi) \rightarrow (V, V^+, \Psi)$ is a differentiable equivariant map extending f and f' .

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The disjoint union of maps makes the set of equivalence classes into an abelian group (Z_2 vector space) denoted $\mathfrak{N}_{m,n}^G(\mathcal{F}, \mathcal{F}')$.

All of the general structure of G actions with families may be carried through for maps. Thus if $j: (\mathcal{F}_0, \mathcal{F}'_0) \subset (\mathcal{F}, \mathcal{F}')$ one has an induced homomorphism

$$j_*: \mathfrak{N}_{m,n}^G(\mathcal{F}_0, \mathcal{F}'_0) \rightarrow \mathfrak{N}_{m,n}^G(\mathcal{F}, \mathcal{F}')$$

by weakening family restrictions, and one has a boundary homomorphism

$$\partial: \mathfrak{N}_{m,n}^G(\mathcal{F}, \mathcal{F}') \rightarrow \mathfrak{N}_{m-1,n-1}^G(\mathcal{F}', \phi)$$

given by $\mathcal{X}(M, \psi), (N, \phi), f = ((\partial M, \psi|_{G \times \partial M}), (\partial N, \phi|_{G \times \partial N}), f|_{\partial N})$.

One easily obtains: If $\mathcal{F}'' \subset \mathcal{F}' \subset \mathcal{F}$ are families in G , the sequence

$$\begin{array}{ccc} \mathfrak{N}_{*,*}^G(\mathcal{F}', \mathcal{F}'') & \xrightarrow{j_{1*}} & \mathfrak{N}_{*,*}^G(\mathcal{F}, \mathcal{F}'') \\ i_{3*} \swarrow & & \searrow i_{2*} \\ & \mathfrak{N}_{*,*}^G(\mathcal{F}, \mathcal{F}') & \end{array}$$

is exact, where j_i are the inclusions.

3. The case $G = Z_2$. For the group Z_2 , there are three families, $\mathcal{A}ll = \{Z_2, \{1\}\}$, $\mathcal{F}_{ree} = \{\{1\}\}$, and ϕ , and hence three nontrivial pairs of families related by the exact sequence

$$\begin{array}{ccc} \mathfrak{N}_{*,*}^{Z_2}(\mathcal{A}ll, \phi) & \xrightarrow{i_*} & \mathfrak{N}_{*,*}^{Z_2}(\mathcal{A}ll, \mathcal{F}_{ree}) \\ k_* \swarrow & & \searrow j_* \\ & \mathfrak{N}_{*,*}^{Z_2}(\mathcal{F}_{ree}, \phi) & \end{array}$$

Note. An ϕ -free action is interpreted as an action on an empty manifold.

Beginning with the free case, one has

Proposition 3.1. $\mathfrak{N}_{m,n}^{Z_2}(\mathcal{F}_{ree}, \phi)$ is isomorphic to the bordism group $\mathfrak{N}_m(\Omega^\infty TBO_{\infty+m-n} \times BZ_2)$, where $\Omega^\infty TBO_{\infty+m-n}$ is a representing space for $(m-n)$ -dimensional cobordism. Further, k_* is zero since there is a splitting $D: \mathfrak{N}_{*,*}^{Z_2}(\mathcal{F}_{ree}, \phi) \rightarrow \mathfrak{N}_{*,*}^{Z_2}(\mathcal{A}ll, \mathcal{F}_{ree})$ assigning to $((M, \psi), (N, \phi), f)$ the induced map $((\bar{M}, \bar{\psi}), (\bar{N}, \bar{\phi}), \bar{f})$ where $\bar{M} = M \times [-1, 1]/(m, t) \sim (\psi(-1, m), -t)$ and $\bar{\psi}(g, (m, t)) = (\psi(g, m), t)$ and $\bar{f}(n, t) = (f(n), t)$.

Proof. Clearly D is a splitting, so $k_* = 0$. To compute $\mathfrak{N}_{*,*}^{Z_2}(\mathcal{F}_{ree}, \phi)$ one defines for $((M, \psi), (N, \phi), f)$ the maps $\tilde{f}: N/Z_2 \rightarrow M/Z_2$ and $\alpha: M/Z_2 \rightarrow BZ_2$ where \tilde{f} is induced by passing to quotients and α classifies the double cover

$M \rightarrow M/Z_2$. As in [2], the map \tilde{f} induces a map into the representing space for cobordism (the limit over the spectral maps of $\Omega^r TBO_{r+m-n}$), which will be denoted $\hat{f}: M/Z_2 \rightarrow \Omega^\infty TBO_{\infty+m-n}$. One associates to $((M, \psi), (N, \phi), f)$ the bordism class of $(M/Z_2, \hat{f} \times \alpha)$. \square

Turning to $\mathcal{R}_{m,n}^{Z_2}(\mathcal{U}, \mathcal{F}_{rel})$, consider an equivariant bordism element $((M, S), (N, T), f)$ where S and T are involutions defining the Z_2 actions.

The fixed point set of S on M , F_S , is the disjoint union of closed submanifolds F_S^k , of dimension k , imbedded in the interior of M . F_S^k has normal bundle ν_{m-k} (of fiber dimension $m-k$) and one may equivariantly identify a collection of disjoint tubular neighborhoods of the sets F_S^k with the disc bundles $D(\nu_{m-k})$ on which S acts as multiplication by -1 .

Similarly, the fixed point set of T on N , F_T , is the disjoint union of closed submanifolds $F_T^{j,k}$, the j -dimensional part which is mapped into F_S^k by f , imbedded in the interior of N , with normal bundle $\tilde{\nu}_{n-j}^k$ (of fiber dimension $n-j$). Let $\nu: F_T^{j,k} \rightarrow BO_{n-j}$ classify $\tilde{\nu}_{n-j}^k$ and consider the diagram

$$\begin{array}{ccc} F_T^{j,k} & \xrightarrow{f} & F_S^k \\ \nu \downarrow & & \\ BO_{n-j} & & \end{array}$$

the "map" bordism class of which is obviously an invariant of the class f . By slight modification of the procedure in [2] this map is realized by a map

$$\hat{f}_j: F_S^k \rightarrow \Omega^\infty(TBO_{\infty+k-j} \wedge BO_{n-j}^+).$$

Specifically, let $i: F_T^{j,k} \rightarrow S^r$ (not hitting the base point) be an imbedding giving an imbedding $i \times f: F_T^{j,k} \rightarrow S^r \times F_S^k$ and hence a collapse $S^r \times F_S^k \rightarrow T\nu_{i \times f}$ onto the Thom space of the normal bundle of the imbedding. As in the Pontrjagin-Thom construction for bordism, $\nu_{i \times f} \times \nu: F_T^{j,k} \rightarrow BO_{r+k-j} \times BO_{n-j}$ Thomifies to a map $T\nu_{i \times f} \rightarrow TBO_{r+k-j} \wedge BO_{n-j}^+$. By pushing off the sphere factor in the composite

$$S^r \times F_S^k \rightarrow T\nu_{i \times f} \rightarrow TBO_{r+k-j} \wedge BO_{n-j}^+$$

and letting r go to infinity, one has the map \hat{f}_j .

Thus the fixed point set of T gives a map

$$\hat{f}_T = \prod_{j \neq n-1} \hat{f}_j: F_S^k \rightarrow \prod_{j \neq n-1} \Omega^\infty(TBO_{\infty+k-j} \wedge BO_{n-j}^+)$$

whose bordism class depends only on the class of f , ignoring the term $j = n-1$.

In addition, another invariant exists. The involution S is a free action on a neighborhood of the sphere bundles $S(\nu_{m-k}) = \partial D(\nu_{m-k})$ and by a small equi-

variant deformation of f , one may suppose f is transverse regular to $\bigcup_k S(\nu_{m-k})$. (If preferred, $\bigcup_k \{D(\nu_{m-k}) - F_S^k\} \rightarrow (0, 1]$ by means of the radial distance and the composite with f has a regular value $r \in (0, 1)$, so that f is transverse regular to $r(\bigcup_k S(\nu_{m-k}))$ for some r .) Then $f^{-1}(S(\nu_{m-k})) = P^{n-1,k}$ is a closed submanifold of N on which T acts freely with $f: (P^{n-1,k}, T) \rightarrow (S(\nu_{m-k}), -1)$ being an equivariant map. Thus one has a diagram

$$\begin{array}{ccc} P^{n-1,k} & \xrightarrow{f} & S(\nu_{m-k}) \\ & & \downarrow \pi \\ & & F_S^k \end{array}$$

and the "map" bordism class of this is an invariant of f .

Techniques for handling such objects are discussed in [4]. Briefly, one may proceed as follows. Passing to orbits of the Z_2 action gives $\bar{f}: P/Z_2 \rightarrow RP(\nu_{m-k})$, the projective space bundle of ν_{m-k} , or $\bar{f}: RP(\nu_{m-k}) \rightarrow \Omega^\infty TBO_{\infty+m-n}$ $((m-1) - (n-1) = m-n)$. Letting $\pi: U \rightarrow F_S^k$ be the principal bundle of ν_{m-k} , one has

$$\bar{f}: U \times_{O_{m-k}} RP(m-k-1) \rightarrow \Omega^\infty TBO_{\infty+m-n}$$

which may be interpreted as on O_{m-k} equivariant map

$$U \rightarrow (\Omega^\infty TBO_{\infty+m-n})^{RP(m-k-1)}.$$

Thinking of this as a free O_{m-k} bordism element one applies the standard technique of crossing with the universal free O_{m-k} space and dividing out the group action to obtain

$$f_P: F_S^k = U/O_{m-k} \rightarrow \{(\Omega^\infty TBO_{\infty+m-n})^{RP(m-k-1)} \times EO_{m-k}\}/O_{m-k}.$$

Note. Projecting this onto $EO_{m-k}/O_{m-k} = BO_{m-k}$ classifies ν_{m-k} , so that one has lifted a classifying map for ν_{m-k} to the associated bundle with fiber $(\Omega^\infty TBO_{\infty+m-n})^{RP(m-k-1)}$. For $k = m-1$, $RP(0)$ is a point and $f_P: F_S^{m-1} \rightarrow \Omega^\infty TBO_{\infty+m-n} \times BO_1$.

One then has

Proposition 3.2. *Assigning to $((M, S), (N, T), f)$ the classes $(F_S^k, \hat{f}_T \times f_P)$ defines an isomorphism of $\mathfrak{N}_{m,n}^{Z_2}(\mathcal{A}, \mathcal{F}_{\text{free}})$ with*

$$\bigoplus_{k=0}^m \mathfrak{N}_k \left(\left\{ \prod_{j \neq n-1} \Omega^\infty(TBO_{\infty+k-j} \wedge BO_{n-j}^+) \right\} \times \left\{ \frac{(\Omega^\infty TBO_{\infty+m-n})^{RP(m-k-1)} \times EO_{m-k}}{O_{m-k}} \right\} \right).$$

Proof. Being given a k -dimensional bordism class $(F, \hat{f}_T \times f_P)$ in this product space is equivalent to being given diagrams

$$\begin{array}{ccc} P^{n-1} & \xrightarrow{h} & S(\xi_{m-k}) \\ & & \downarrow \pi \\ & & F \end{array}$$

where h is an equivariant map of Z_2 spaces, ξ_{m-k} being an $m-k$ plane bundle, and

$$\begin{array}{ccc} F^j & \xrightarrow{h_j} & F \\ \rho \downarrow & & \\ BO_{n-j} & & \end{array}$$

where ρ classifies an $n-j$ plane bundle ρ_{n-j} over F^j , $j \neq n-1$. Combining these, one has an $(\mathcal{A}ll, \mathcal{F}_{ree})$ m -dimensional manifold $D(\xi_{m-k})$ with involution -1 in the fibers together with an n -dimensional $(\mathcal{A}ll, \mathcal{F}_{ree})$ manifold which is the union of $P \times [-1, 1]/(p, t) \sim (Tp, -t)$ with involution $T \times 1$ (where T is the involution on P) and the manifolds $D(\rho_{n-j})/(x \sim -x; x \in S(\rho_{n-j}))$ with involution induced by -1 in the fibers, together with an equivariant map into $D(\xi_{m-k})$ given by sending $(p, t) \in P \times [-1, 1]$ to $t \cdot h(p)$ and given by $h_j \circ \pi_j$ on $D(\rho_{n-j})$, where π_j is the projection. This constructs an inverse Q to the homomorphism $R: \mathcal{N}_{m,n}^{Z_2}(\mathcal{A}ll, \mathcal{F}_{ree}) \rightarrow A$, where A is the given direct sum.

By taking fixed sets and applying transverse regularity, $RQ = 1$ trivially. To see that $QR = 1$, one considers the bordism element $((M, S), (N, T), f)$. After deformation of f as before, one may suppose f is transverse regular to $\bigcup_k S(\nu_{m-k})$ with $f^{-1}(S(\nu_{m-k})) = P^{n-1,k}$. Then one has $f: f^{-1}(\bigcup_k D(\nu_{m-k})) \rightarrow \bigcup_k D(\nu_{m-k})$ imbedded in $f: N \rightarrow M$ with the complementary part being free as Z_2 map, so one may suppose $M = \bigcup_k D(\nu_{m-k})$ by excising the complementary portion. By radial deformation in the fibers of $D(\nu_{m-k})$, one may suppose that $P^{n-1,k} = \partial(f^{-1}(D(\nu_{m-k})))$ maps to $\partial D(\nu_{m-k})$ with a tubular neighborhood $P^{n-1,k} \times [1, 0]$ of $P^{n-1,k}$ mapping by radial extension into $D(\nu_{m-k})$ ($f(p, t) = t \cdot f(p)$) and with the remainder of $f^{-1}(D(\nu_{m-k}))$ being mapped into the zero section F_S^k . By splitting $f^{-1}(D(\nu_{m-k}))$ along $P \times 0$ and dividing out the free involution T on the two copies of P introduced, one replaces f by the map of $P \times [-1, 1]/(p, t) \sim (Tp, -t)$ into $D(\nu_{m-k})$ given as part of QR together with a map of a closed Z_2 manifold L^n into F_S^k . This splitting may be accomplished via an equivalence: specifically one may suppose f maps a neighborhood $P \times [1, -\epsilon]$ of P by $f(p, t) = tf(p)$ if $t \geq 0$ and $f(p, t) = 0 \cdot f(p)$ if $t \leq 0$ by a small deformation, and letting $V = f^{-1}(D(\nu_{m-k})) \times [0, 1]$ with $(p, t, 1) \sim (Tp, t, 1)$ for $t \in [0, -\epsilon]$ with

map $f \cdot \pi$ into $D(\nu_{m-k})$ gives an equivalence of $f^{-1}(D(\nu_{m-k})) \times 0$ with the split map at the $(\times 1)$ -edge. The splitting along $P \times 0$ gives L a fixed submanifold of codimension one, so that the maps $L \rightarrow F_S^k$ and $\bigcup_{j \neq n-1} D(\rho_{n-j})/(x \sim -x; x \in S(\rho_{n-j})) \rightarrow F_S^k$ which occurs in the image of QR are Z_2 bordism elements of the fixed space F_S^k having the same fixed data of codimension not equal to one, hence are bordant in F_S^k . Combining all of these constructions gives an equivalence of $((M, S), (N, T), f)$ and its image under QR . \square

One may completely analyze the exact sequence for Z_2 equivariant maps in terms of the decomposition given by Proposition 3.2. Applying the splitting D of Proposition 3.1, one may identify

$$D: \mathfrak{N}_{m-1, n-1}^{Z_2}(\mathcal{F}_{ree}, \phi) \rightarrow \mathfrak{N}_{m, n}^{Z_2}(\mathcal{A}ll, \mathcal{F}_{ree})$$

with the homomorphism

$$\begin{aligned} & \mathfrak{N}_{m-1}(\Omega^\infty TBO_{\infty+m-n} \times BO_1) \\ & \rightarrow \mathfrak{N}_{m-1} \left(\left\{ \prod_{j \neq n-1} \Omega^\infty(TBO_{\infty+m-1-j} \wedge BO_{n-j}^+) \right\} \times \Omega^\infty TBO_{\beta+m-n} \times BO_1 \right) \end{aligned}$$

induced by the slice at a point in the first factor. Corresponding to this, one has a splitting $P: \mathfrak{N}_{m, n}^{Z_2}(\mathcal{A}ll, \mathcal{F}_{ree}) \rightarrow \mathfrak{N}_{m, n}^{Z_2}(\mathcal{A}ll, \phi)$ assigning to $((M, S), (N, T), f)$ the class obtained as follows. Let f be made transverse regular to $\bigcup_k S(\nu_{m-k})$ and consider the map

$$\begin{aligned} \tilde{f}: \bigcup_k f^{-1}(D(\nu_{m-k}))/ (x \sim Tx; x \in \partial f^{-1}(D(\nu_{m-k}))) \\ \rightarrow \bigcup_k D(\nu_{m-k})/ (x \sim Sx; x \in S(\nu_{m-k})) \end{aligned}$$

induced by f . $P((M, S), (N, T), f)$ is given by the class of \tilde{f} . Clearly $i_* P(x)$ and x have the same fixed data except for the portion of the fixed set $\bigcup_k S(\nu_{m-k})/Z_2$ introduced, which is codimension one, with the inverse image fixed set being $\bigcup_k \partial f^{-1}(D(\nu_{m-k}))/Z_2$ which is also codimension one. The new invariants introduced are all given by the transverse regularity construction as the maps

$$\begin{aligned} \partial f^{-1}(D(\nu_{m-k})) & \rightarrow S(\nu_{m-k}) \\ & \downarrow \\ & S(\nu_{m-k})/Z_2 \end{aligned}$$

which are the classes $D(f: \partial f^{-1}(D(\nu_{m-k})) \rightarrow S(\nu_{m-k}))$. Thus $i_* Px = x \bmod \text{image } D$. Further, $\text{image } D \subset \text{kernel } P$, for being given a free map $g: (A, T) \rightarrow (B, S)$, $PD(g)$ is the map

$$\begin{aligned} \overline{g \times 1}: A \times [-1, 1] / \{(a, t) \sim (Ta, t), t = \pm 1\} \\ \rightarrow B \times [-1, 1] / \{(b, t) \sim (Sb, t), t = \pm 1\} \end{aligned}$$

with involutions $T \times 1$ and $S \times 1$. Letting $g': \bar{A} \rightarrow \bar{B}$ be the map constructed for D , $\overline{g \times 1}$ is the boundary of the map $g' \times 1: \bar{A} \times [0, 1] \rightarrow \bar{B} \times [0, 1]$. The sequence

$$0 \rightarrow \mathfrak{N}_{*,*}^{Z_2}(\mathcal{F}_{ree}, \phi) \xrightarrow{D} \mathfrak{N}_{*,*}^{Z_2}(\mathcal{Qll}, \mathcal{F}_{ree}) \xrightarrow{P} \mathfrak{N}_{*,*}^{Z_2}(\mathcal{Qll}, \phi) \rightarrow 0$$

is then exact with D splitting $j_*\partial$ and P splitting i_* .

Thus, one sees that Z_2 equivariant maps may be analyzed in a fashion which is pure analogy with the Conner and Floyd analysis of Z_2 actions in [1, §28].

4. The case $G = Z_p$, p odd. Let p be an odd prime and consider $G = Z_p$. Again there are three families: \mathcal{Qll} , \mathcal{F}_{ree} , and ϕ , giving three nontrivial pairs of families related by the exact sequence

$$\begin{array}{ccc} \mathfrak{N}_{*,*}^{Z_p}(\mathcal{F}_{ree}, \phi) & \xrightarrow{k_*} & \mathfrak{N}_{*,*}^{Z_p}(\mathcal{Qll}, \phi) \\ & \searrow j_*\partial & \nearrow i_* \\ & \mathfrak{N}_{*,*}^{Z_p}(\mathcal{Qll}, \mathcal{F}_{ree}) & \end{array}$$

As with the Z_2 case one has

Proposition 4.1. Assigning to $((M, \psi), (N, \phi), f)$ the diagram $N/Z_p \xrightarrow{\bar{f}} M/Z_p \xrightarrow{\alpha} BZ_p$, where α classifies the cover $M \rightarrow M/Z_p$ defines an isomorphism of $\mathfrak{N}_{m,n}^{Z_p}(\mathcal{F}_{ree}, \phi)$ with $\mathfrak{N}_m(\Omega^\infty TBO_{\infty+m-n} \times BZ_p)$.

Before beginning the analysis of the $(\mathcal{Qll}, \mathcal{F}_{ree})$ case recall that there is a classifying space $F'_{Z_p}(BO_k)$ for k -plane bundles with linear Z_p action over Z_p fixed spaces having the property that only the zero section is pointwise fixed by Z_p ; i.e. if $\xi \rightarrow X$ is a k -plane bundle with Z_p action over the Z_p fixed space X and no fiber in ξ contains a copy of the trivial representation, then ξ is classified by a map into $F'_{Z_p}(BO_k)$. From [1, §38], one has $F'_{Z_p}(BO_{2k}) = \bigcup BU_{k_1} \times \cdots \times BU_{k_{(p-1)/2}}$, the union being for $k_1 + \cdots + k_{(p-1)/2} = k$, and $F'_{Z_p}(BO_{2k+1}) = \phi$.

Now consider a class $((M, \psi), (N, \phi), f)$ in $\mathfrak{N}_{m,n}^{Z_p}(\mathcal{Qll}, \mathcal{F}_{ree})$. The fixed set $F_{Z_p}(M)$ is a union of closed submanifolds $F_{Z_p}^{m-2k}(M)$ of codimension $2k$ imbedded in the interior of M with normal bundle ν_{2k} classified by a map $\nu_{2k}: F_{Z_p}^{m-2k}(M) \rightarrow F'_{Z_p}(BO_{2k})$, and with a collection of disjoint tubular neighborhoods of the sets $F_{Z_p}^{m-2k}(M)$ being equivariantly identified with the disc bundles $D(\nu_{2k})$.

Similarly, the fixed set of ϕ on N is a disjoint union of closed submanifolds

$F_{Z_p}^{n-2j,k}(N)$ of codimension $2j$ imbedded in the interior of N and mapping into $F_{Z_p}^{m-2k}(M)$ under f . Letting $\tilde{\nu}_{2j}^k$ denote the normal bundle of $F_{Z_p}^{n-2j,k}(N)$, one has a diagram

$$\begin{array}{ccc} F_{Z_p}^{n-2j,k}(N) & \xrightarrow{f} & F_{Z_p}^{m-2k}(M) \\ \tilde{\nu} \downarrow & & \\ F'_{Z_p}(BO_{2j}) & & \end{array}$$

inducing a map

$$\hat{f}_j: F_{Z_p}^{m-2k}(M) \rightarrow \Omega^\infty(TBO_{\infty+m+2j-n-2k} \wedge F'_{Z_p}(BO_{2j})^+).$$

One may then identify disjoint tubular neighborhoods of the sets $F_{Z_p}^{n-2j,k}(N)$ with the disc bundles $D(\tilde{\nu}_{2j}^k)$ and by deforming radially in the fibers of these discs, may assume $f|_{D(\tilde{\nu}_{2j}^k)} = f|_{F_{Z_p}^{n-2j,k}(N)} \circ \pi$, where π is the projection. One may then cut N along the sets $S(\tilde{\nu}_{2j}^k)$ and identify the resulting boundaries under the antipodal map (which commutes with the Z_p action by bundle maps and is compatible with f since f factors through the projection). This replaces $f: N \rightarrow M$ by an equivalent map $f': N' \cup \bigcup_{k,j} D(\tilde{\nu}_{2j}^k)/(x \sim -x; x \in S(\tilde{\nu}_{2j}^k)) \rightarrow M$ where $f'(\partial N') \subset \partial M$ and the Z_p action on N' , call it ϕ' , is a free action.

Since (N', ϕ') is free, one may deform f' to be transverse regular to $\bigcup_k F_{Z_p}^{m-2k}(M)$ with $f'^{-1}(F_{Z_p}^{m-2k}(M)) = \bar{N}^{n-2k}$ being a free Z_p manifold with action ϕ' , and with normal bundle $\bar{\nu}_{2k}$ being induced from ν_{2k} . By a further deformation of f' , one may suppose f' is given on the collection of disjoint tubular neighborhoods $D(\bar{\nu}_{2k})$ by the bundle maps covering $f'|_{\bar{N}^{n-2k}}$.

Then $f': N' \cup \bigcup D(\tilde{\nu}_{2j}^k)/(Z_2|S(\tilde{\nu}_{2j}^k)) \rightarrow M$ contains the map $f': \bigcup D(\bar{\nu}_{2k}) \cup \bigcup D(\tilde{\nu}_{2j}^k)/(Z_2|S(\tilde{\nu}_{2j}^k)) \rightarrow \bigcup D(\nu_{2k})$ and the complement is a free Z_p map, which may be excised.

Corresponding to the diagram

$$\begin{array}{ccc} \bar{N}^{n-2k}/Z_p & \xrightarrow{\tilde{f}'} & F_{Z_p}^{m-2k}(M) \\ \alpha_k \downarrow & & \\ BZ_p & & \end{array}$$

where α_k classifies the cover by \bar{N} and \tilde{f}' is induced by f' , one has

$$\beta_k: F_{Z_p}^{m-2k}(M) \rightarrow \Omega^\infty(TBO_{\infty+m-n} \wedge BZ_p^+).$$

Proposition 4.2. Assigning to $((M, \psi), (N, \phi), f)$ the classes $(F_{Z_p}^{m-2k}(M), \nu_{2k} \times (\times_j \hat{f}_j) \times \beta_k)$ gives an isomorphism Θ of $\mathfrak{R}_{m,n}^{Z_p}(\mathcal{A}\mathcal{L}, \mathcal{F}_{ree})$ with

$$\bigoplus_{k=0}^{m/2} \mathfrak{R}_{m-2k} \left(F_{Z_p}'(BO_{2k}) \times \left\{ \times_j \Omega^\infty(TBO_{\infty+m+2j-n-2k} \wedge F_{Z_p}'(BO_{2j})^+) \right\} \right. \\ \left. \times \Omega^\infty(TBO_{\infty+m-n} \wedge BZ_p^+) \right).$$

Rather than brutalize the entire thing, let A denote this sum, and define a map $P: A \rightarrow \mathfrak{R}_{m,n}^{Z_p}(\mathcal{A}\mathcal{L}, \phi)$ as follows. Being given $\alpha = (F^{m-2k}, \nu \times (\times_j \hat{f}_j) \times \beta)$ in A , one has diagrams $\bar{N}^{n-2k} \xrightarrow{f'} F^{m-2k}$ with \bar{N} a free Z_p space, and

$$\begin{array}{ccc} H^{n-2j} & \xrightarrow{f} & F^{m-2k} \\ \rho \downarrow & & \\ F_{Z_p}'(BO_{2j}) & & \end{array}$$

with ρ inducing a bundle ρ_{2j}^k with Z_p action over H^{n-2j} , and a bundle ξ_{2k} over F^{m-2k} with Z_p action, induced by the map ν . One may then form the Z_p manifold $M' = D(\xi_{2k})/(x \sim -x \mid x \in S(\xi_{2k}))$ with Z_p action given by the bundle action, and the manifold $N' = D(f'^*\xi_{2k})/(x \sim -x \mid x \in S(f'^*\xi_{2k}))$ with induced bundle Z_p action and map $f': N' \rightarrow M'$ arising from the map of the induced bundle. One lets $E_j = D(\rho_{2j}^k)/(x \sim -x \mid x \in S(\rho_{2j}^k))$ with bundle Z_p action and map $f': E_j \rightarrow M'$ given by $f \circ \pi$, π being the projection on H^{n-2j} .

Let $p(\alpha) = (M', N' \cup \bigcup_j E_j, f')$. From all of the above constructions one clearly has $\Theta i_* P(\alpha) = \alpha$ and have shown the equivalence of $i_* P(\Theta(M, N, f))$ and (M, N, f) . Thus Θ is an isomorphism with inverse $i_* P$.

Finally, $P\Theta$ defines a splitting for i_* and one has

Proposition 4.3. The exact sequence for Z_p families splits and one has

$$\mathfrak{R}_{m,n}^{Z_p}(\mathcal{A}\mathcal{L}, \phi) \cong \mathfrak{R}_{m,n}^{Z_p}(\mathcal{F}_{ree}, \phi) \oplus \mathfrak{R}_{m,n}^{Z_p}(\mathcal{A}\mathcal{L}, \mathcal{F}_{ree}).$$

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